v. 90

QNo→ State and Prove the Maximum Modulus Principle.

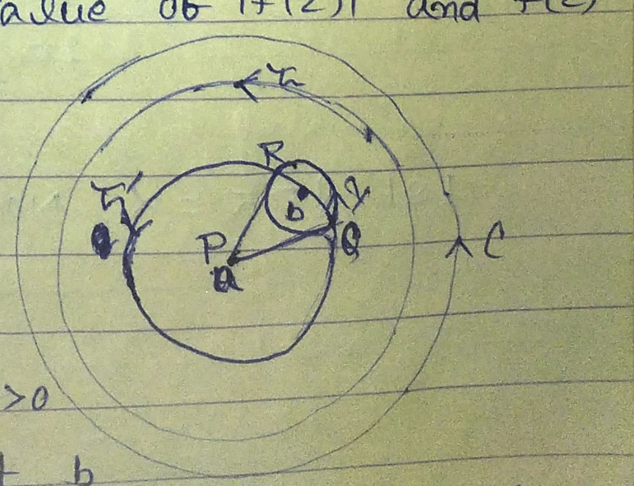
Ans→ **Statement:-** Let $f(z)$ be analytic within and on a simple closed contour C. Then $|f(z)|$ reaches its maximum value on C and not inside C, unless $f(z)$ is a constant.

In other words, if M is the maximum value of $|f(z)|$ on and within C, then unless f is a constt., $|f(z)| < M$ for every point z within C.

Proof:- Since $f(z)$ is analytic it will be continuous. Therefore, it must attained its maximum value M at some point on or within C. Let $f(z)$ is not a constant, then we shall have to show ~~tat~~ that $|f(z)|$ takes the value M at some point on C. If possible, this value is not attained on the boundary C. $\therefore |f(a)| = M$.

Let γ be a circle inside C with centre at a.

$\because M$ is the maximum value of $|f(z)|$ and $f(z)$ is not a constant, there must exist a point, say b, inside γ such that $|f(b)| < M$.



Let $|f(b)| = M - \epsilon$, where $\epsilon > 0$

$\because |f(z)|$ is continuous at b

for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$|f(z)| - |f(b)| < \frac{\epsilon}{2}$ for all z satisfying $|z-b| <$

$\therefore |f(z)| - |f(b)| \leq ||f(z)| - |f(b)|| < \frac{\epsilon}{2}$ $\forall z$

satisfying $|z-b| < \delta$.

$\therefore |f(z)| < |f(b)| + \frac{\epsilon}{2} = M - \epsilon + \frac{\epsilon}{2} = M - \frac{\epsilon}{2}$, $\forall z$

satisfying $|z-b| < \delta$.

We draw a circle Υ' with centre at a ~~and~~ such that it passes through b. The arc QR of the circle Υ' lies within γ, where γ is circle with centre b and radius δ.

Therefore, $|f(z)| < M - \frac{\epsilon}{2}$ for points z on the QR on the remaining arc of Υ', we have

$$f(a) = \frac{1}{2\pi i} \int_{\Upsilon'} \frac{f(z)}{z-a} dz.$$

Let $z - a = re^{i\theta}$ $\therefore dz = rie^{i\theta} d\theta$.

$$\therefore f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot rie^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta}) d\theta.$$

Let $QPR = \alpha$, Now, $f(a) = \frac{1}{2\pi} \int_0^{\alpha} f(a+re^{i\theta}) d\theta$

$$+ \frac{1}{2\pi} \int_{\alpha}^{2\pi} f(a+re^{i\theta}) d\theta.$$

$\therefore |f(a)| \leq \frac{1}{2\pi}\left|\int_0^{\alpha} f(a+re^{i\theta})\,d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} f(a+re^{i\theta})\,d\theta\right|$

$\leq \frac{1}{2\pi}\int_0^{\alpha} |f(a+re^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} |f(a+re^{i\theta})|\,d\theta$

$< \frac{1}{2\pi}(M-\epsilon)\int_0^{\alpha} d\theta + \frac{M}{2\pi}\int_{\alpha}^{2\pi} d\theta.$

$= \frac{M-\epsilon}{2\pi}\cdot\alpha + \frac{M}{2\pi}(2\pi-\alpha)$

$= M - \frac{\alpha\epsilon}{2\pi}$

i.e. $M = |f(a)| < M - \frac{\alpha\epsilon}{2\pi}$ which is imbossible.

Hence the theorem ballows ~~ie~~. that is $|f(z)|$ ~~is~~ Cannot reaches its maximum value on the boundary of C.